§1 February 7, 2022

§1.1 AIME 2002 I Writeups

We were given this as a practice in math club.

Problem 1.1 (AIME I 2002/11)
Let $ABCD$ and $BCFG$ be two faces of a cube with $AB = 12$. A beam of light emanates from vertex A and reflects off face $BCFG$ at point P, which is 7 units from $\overline{BG}$ and 5 units from $\overline{BC}$. The beam continues to be reflected off the faces of the cube. The length of the light path from the time it leaves point A until it next reaches a vertex of the cube is given by $m\sqrt{n}$, where m and n are integers and n is not divisible by the square of any prime. Find $m + n$.

Solution. If you are familiar with physics at all, the idea of reflection should bring something up in your head. Remember how we are just seeing something through the mirror? With this problem it really is just the same thing.

Admittedly, this is a trick that I don't think would immediately come to your mind, but it was the obvious one to me.

Consider the vector $(12, 7, 5)$ starting at the corner of the box, now imagine it just passes through the box. To find where it hits a corner, we just continue to tile the box out. If the vector lands on a coordinate whose components are all multiples of 12, then it lands on the corner of the box.

Clearly, we need 12 vectors for all coordinates to be multiples of 12. The calculation for the length of the light's path is then:

$$m\sqrt{n} = 12 \cdot \sqrt{12^2 + 5^2 + 7^2} = 12\sqrt{218} \implies m + n = \boxed{230}.$$

□

Problem 1.2 (AIME I 2002/13)
In triangle ABC the medians $\overline{AD}$ and $\overline{CE}$ have lengths 18 and 27, respectively, and $AB = 24$. Extend $\overline{CE}$ to intersect the circumcircle of ABC at F. The area of triangle AFB is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Solution. Let I be the intersections of the medians. We know from median properties that $AI = 12, ID = 6$, and $CI = 18, IE = 9$.

Claim 1.3 — $\triangle AEC \sim \triangle FEB$.

Proof. $\angle AEC = \angle BEF$ since they are opposite angles. $\angle ACE = \angle EBF$ since they inscribe the same arc in the circle. ■

Similar reasoning follows to show that $\triangle CEB \sim \triangle AEF$.

Moreover, since the short sides of $\triangle AEF$ and $\triangle FEB$ and their respective similar triangles are both equal, we know that their areas are proportional by the same amount.

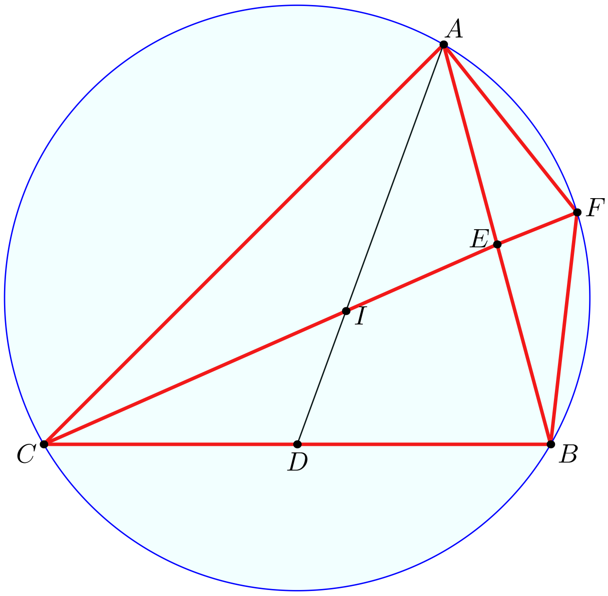

Figure 1: Problem 13

So really all we need to do is find $[ABC]$ and find some constant between it and $[AFB]$. Let $\angle AEC = \theta$, then

$$\begin{aligned}[ABC] &= [AEC] + [BEC] \\ &= \frac{1}{2} \cdot 12 \cdot 27 \sin\theta + \frac{1}{2} \cdot 12 \cdot 27 \sin 180 - \theta \\ &= 12 \cdot 27 \sin\theta.\end{aligned}$$

Well now we're stuck... what's $\sin\theta$? Well we know all the lengths of $\triangle AEI$, and it contains the angle we are looking for, which motivates trying to find $\cos\theta$ using law of cosines. It turns out that $\cos\theta = \frac{3}{8}$, which means $\sin\theta = \frac{\sqrt{55}}{8}$. Therefore $[ABC] = \frac{81\sqrt{55}}{2}$. Now the proportion between $\triangle BEF$ and $\triangle AEC$ is

$$\frac{CE}{AE} = \frac{BE}{EF} \implies \frac{27}{12} = \frac{12}{EF} \implies EF = \frac{16}{3}.$$

Then we can find the proportion with,

$$\frac{EF}{AE} = \frac{4}{9}.$$

Therefore the area is scaled by $(\frac{4}{9})^2 = \frac{16}{81}$. It is easy to see that this is also true for $\triangle AEB$ and $\triangle AEF$.

Finally then,

$$m\sqrt{n} = [AFB] = \frac{16}{81} \cdot [ABC] = \frac{16}{81}\frac{81\sqrt{55}}{2} = 8\sqrt{55} \implies m + n = \boxed{063}.$$

□

Problem 1.4 (AIME I 2002/14)

A set $\mathcal{S}$ of distinct positive integers has the following property: for every integer x in $\mathcal{S}$, the arithmetic mean of the set of values obtained by deleting x from $\mathcal{S}$ is an integer. Given that 1 belongs to $\mathcal{S}$ and that 2002 is the largest element of $\mathcal{S}$, what is the greatest number of elements that $\mathcal{S}$ can have?

Solution. Let t be the sum of all elements of $\mathcal{S}$. From the problem statement, we know $n-1 \mid t-1$. Therefore

$$t \equiv 1 \pmod{n-1}.$$

But also $n-1 \mid t-2002$, so

$$t \equiv 2002 \pmod{n-1}.$$

Therefore we find that

$$2001 \equiv 0 \pmod{n-1} \implies n-1 \mid 2001.$$

Claim 1.5 — All the elements must be congruent modulo $n-1$, where n is the number of elements in $\mathcal{S}$.

Proof. Suppose two elements a_i and a_j are not equivalent modulo $n-1$. Then, as before,

$$t \equiv a_i \pmod{n-1} \text{ and } t \equiv a_j \pmod{n-1},$$

but then

$$a_j - a_i \equiv 0 \pmod{n-1}.$$

Contradiction. ■

Now since $n-1 \mid 2001 = 3 \cdot 23 \cdot 29$, we can reduce our choices even more.

If we can only choose integers congruent modulo $n-1$ between 1 and 2002, we know that there are about $n-1 \mid 2002+1$ elements to choose from. If there are n elements and about $2002/n+1$ elements to choose from, it needs to somewhat satisfy the inequality

$$n + (n-1)n \leq 2002.$$

The solutions are less than 45, so the highest integer divisor of 2002 that works is $n-1 = 29$. Therefore $n = \boxed{030}$.

Remark 1.6. This is a very rough bound, and we are pretty lucky that 2001 has factors obviously below and above our bound.

□