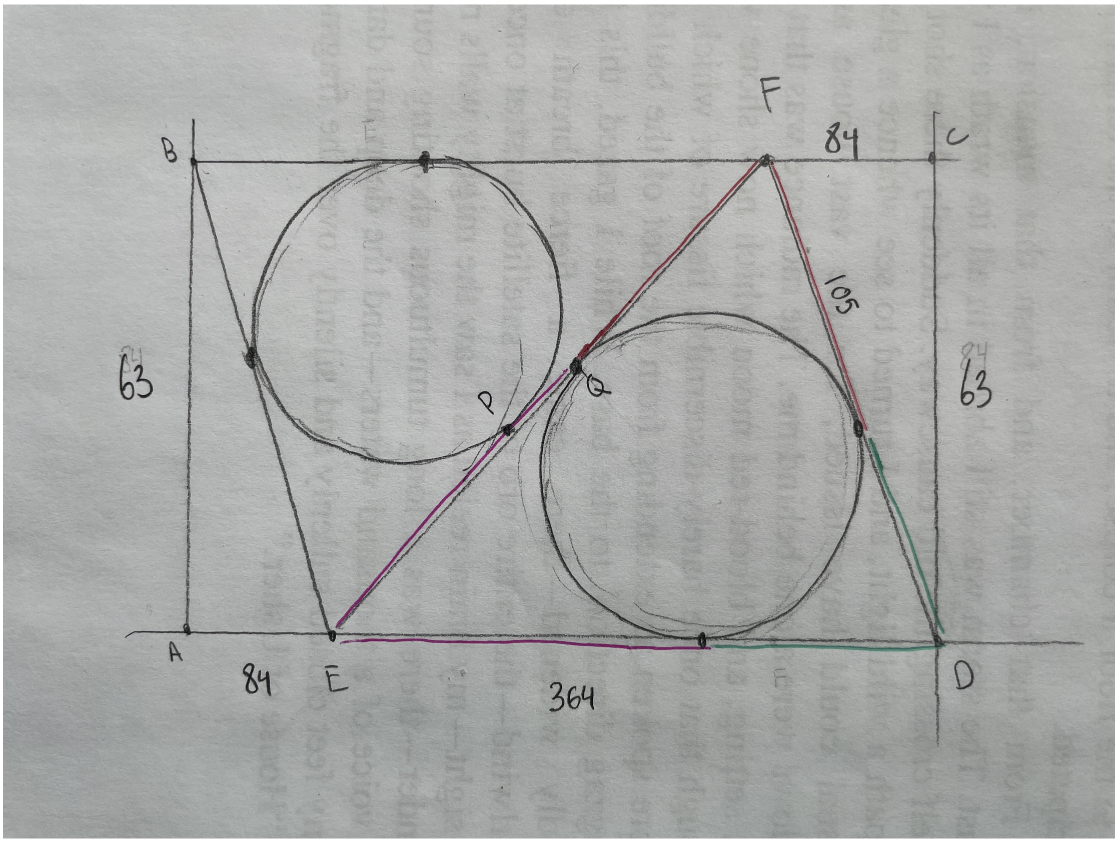

Figure 1: Diagram (not to scale)

§17 January 12, 2022

§17.1 AIME PSET 3

Problem 17.1 (AIME II 2007/9)
Rectangle $ABCD$ is given with $AB = 63$ and $BC = 448$. Points E and F lie on AD and BC respectively, such that $AE = CF = 84$. The inscribed circle of triangle BEF is tangent to EF at point P, and the inscribed circle of triangle DEF is tangent to EF at point Q. Find PQ.

Solution. Firstly, by symmetry, we can just find FQ, and subtract it from EF twice to get PQ. Notice that the colored lines are equal because of incircle properties. We want to find the length of the red segments, which each have length FQ. Letting $EF = s$, we find that

$$s + 105 - 364 = 2FQ \implies FQ = \frac{s - 259}{2}.$$

Then

$$PQ = s - 2FQ = s - (s - 259) = \boxed{259}$$

□

§17.2 NICE Journal 1 - Combinatorics Problems

Problem 17.2 (A(N)IME 2020/6)

The garden of Gardenia has 15 identical tulips, 16 identical roses, and 23 identical sunflowers. David wants to use the flowers in the garden to make a bouquet. However, the Council of Elders has a law that any bouquet using Gardenia's flowers must have more roses than tulips and more sunflowers than roses. IfN is the number of ways David can make his bouquet, what is the remainder when N is divided by 1000?

Solution. Big thing to notice is that the numbers have to be different, and they have to be sorted. This should ring buzzers for choosing numbers and sorting them in a specific, ordered way.

Let's start with some easy case, and see if we can systematically kill the other ones.

If we let flowers be up to 15, which is possible for all. We just have to pick three numbers from 0 to 15. Then assign the smallest to tulips(t), middle to roses(r), largest to sunflowers(s). There are $\binom{16}{3}$ ways to do this.

Next, we can keep tulips and roses below or equal to 15, and let sunflower range above 15. From the first 16 numbers, we choose 2, assigning the lowest to t and the highest to r. The sunflower can be any number from $\{16, 17, \ldots, 23\}$, which gives it 8 choices. There are $\binom{16}{2} \cdot 8$ ways to do this.

Finally, we have the case that $r = 16$. Roses can be anything from 0 to 15, and sunflowers can be anything from 17 to 23. So there are $16 \cdot 7$ ways to do this.

After verifying that this covers everything (it does) we find N is

$$N = \binom{16}{3} + \binom{16}{2} \cdot 8 + 16 \cdot 7 = 1\boxed{632}.$$

□

Problem 17.3 (AIME I 2015/12)

Consider all 1000-element subsets of the set $\{1, 2, 3, \ldots, 2015\}$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{q}{p}$, where p and q are relatively prime positive integers. Find $p + q$.

Solution. We do expected value. Recall that we want to multiply our number by the number of ways to get it, divided by the total number of ways (since we have uniform probability of choosing any set). If 1 is the least element, then there are 2014 other options to choose from the for the remaining 999 elements.

2 means 2013 other options for 999.

3 means 2012 other options for 999.

You get it. The sum we want is

$$\sum_{i=1}^{1016} i \cdot \binom{2015 - i}{999} = 1\binom{2014}{999} + 2\binom{2013}{999} + \cdots + 1016\binom{999}{999}$$

Now time for the summation transformations that make me look insane. Recall the Hockey-Stick identity:

$$\binom{r}{r} + \binom{r+1}{r} + \ldots \binom{m}{r} = \binom{m+1}{r+1}.$$

We can turn the sum into

$$\begin{aligned}\sum_{i=1}^{1016} i \cdot \binom{2015-i}{999} &= \binom{2014}{999} + \binom{2013}{999} + \cdots + \binom{999}{999} \\ &+ \binom{2013}{999} + \binom{2012}{999} + \cdots + \binom{999}{999} \\ &+ \vdots \\ &+ \binom{999}{999} \\ &= \binom{2015}{1000} + \binom{2014}{1000} + \ldots \binom{1000}{1000} \\ &= \binom{2016}{1001}\end{aligned}$$

DOUBLE Hockey-Stick identity!!! Okay so the total number of ways of choosing 1000 elements from our set is $\binom{2015}{1000}$. So the solution is the total sum over the number of cases

$$\frac{p}{q} = \frac{\binom{2016}{1001}}{\binom{2015}{1000}} = \frac{\frac{2016!}{1001! \cdot 1015!}}{\frac{2015!}{1000!1015!}}] = \frac{2016}{1001} = \frac{288}{143} \implies p + q = \boxed{431}.$$

□

Problem 17.4 (AIME II 2006/10)

Seven teams play a soccer tournament in which each team plays every other team exactly once. No ties occur, each team has a 50% chance of winning each game it plays, and the outcomes of the games are independent. In each game, the winner is awarded a point and the loser gets 0 points. The total points are accumilated to decide the ranks of the teams. In the first game of the tournament, team A beats team B. The probability that team A finishes with more points than team B is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Don't try to see a Catalan number solution like I did. We can split the ending of the games in three ways:

$A < B$	$A = B$	$A > B$
Ties/B Wins	A wins	A wins

We only want the last two! By the symmetry of this situation, we know these equations are true:

$$P(\text{same}) + P(A > B) + P(A < B) = 1$$
$$P(A < B) = P(A > B).$$

All we should need to calculate is the number of ways that A and B win the same number of days. We use a special case of Vandermonde's Identity:

$$\sum_{l=0}^{n} \binom{n}{l}^2 = \binom{2n}{n}.$$

(there are ways to do this without the identity, but it seems like a useful one to learn).

So there are

$$\sum_{i=0}^{5} \binom{5}{i}^2 = \binom{10}{5}$$

ways for them to tie. There are 2^{10} possible outcomes. Using the symmetric probabilities from above, we see that

$$P(A > B) = \frac{1}{2}\left(1 - \frac{\binom{10}{5}}{2^{10}}\right) = \frac{386}{1024}$$

So the desired probability of A winning is

$$\frac{m}{n} = P(A > B) + P(\text{same}) = \frac{386}{1024} + \frac{252}{1024} = \frac{638}{1024} = \frac{319}{512} \implies m + n = \boxed{831}.$$

$\square$