§15 January 10, 2021

§15.1 AIME PSET 3

(thank you amctrivial)

Problem 15.1 (AIME I 2018/8)

Let $ABCDEF$ be an equiangular hexagon such that $AB = 6, BC = 8, CD = 10$, and $DE = 12$. Denote by d the diameter of the largest circle that fits inside the hexagon. Find d^2.

Solution. Since $ABCDEF$ is equiangular, that inspires us to extend some of the lines, thus forming a equilateral triangle. Say A' is the intersection of the extension of AF and BC. Repeat for B' and C' with BC, DE, and DE, AF respectively. Then to find the side length of this huge triangle, we recognize that $A'BA$, $CB'D$, and FEC' are all equilateral.

So then we know that $A'B' = 6 + 8 + 10 = 24$. Similarly, since all the triangle's sides are equal, and we now know that $EF = 2$, and $AF = 16$, giving us all the side lengths!

Claim 15.2 — The shortest line that can be drawn inside the hexagon with points on the hexagon is the length of the diameter of the largest circle that can be drawn in the hexagon.

Proof. If we could draw a larger one, then it would certainly pass through the hexagon! So it is not fully inside of it. ■

The shortest line is between $A'C'$ and CD. The length is just the difference of the altitudes of $A'B'C'$ and CDB', which is $d = 7\sqrt{3}$. Then our answer is $d^2 = \boxed{147}$. □

Problem 15.3 (AIME II 2004/3)

A solid rectangular block is formed by gluing together N congruent 1-cm cubes face to face. When the block is viewed so that three of its faces are visible, exactly 231 of the 1-cm cubes cannot be seen. Find the smallest possible value of N.

Solution. If our sides have length a, b, and c, the blocks we cannot see exclude the outside layer of three sides. Therefore we can clearly see that the blocks we cannot see have dimensions

$$(a-1)(b-1)(c-1) = 231.$$

$231 = 3 \cdot 7 \cdot 11$. We want our numbers to be small, so we keep them factored. We don't even need to test cases because there is only one factorization with 3 primes. Therefore, we let

$$(a-1) = 3 \qquad (b-1) = 7 \qquad (c-1) = 11$$
$$\implies N = abc = 4 \cdot 8 \cdot 12 = \boxed{384}.$$

(semi-troll) □