§14 January 9, 2021

Okay fine here's a problem:

Problem 14.1 (Based on Sweden 2010)
A herd of 1000 cows of nonzero weight is given. Prove that we can remove one cow such that the remaining 999 cows cannot be split into two halves of equal weights.

Solution. Assume that we can! That is, nonzero cow weights exist that satisfy the condition. Then if we remove any cow of weight c_i, the remaining cows should be able to be split into two equal halves. If cows are on one side of the scale we can have them be negated, and the other side we just take the positive value. The sum of these should be zero! This motivates us to create the matrix equation.

$$T \cdot \vec{c} = \vec{0} \tag{14.1}$$

$$\begin{bmatrix} 0 & \pm1 & \pm1 & \cdots & \pm1 \\ \pm1 & 0 & \pm1 & \cdots & \pm1 \\ \pm1 & \pm1 & 0 & \cdots & \pm1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \pm1 & \pm1 & \pm1 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{1000} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{14.2}$$

This seemed pretty hard to show is impossible, but since we are already considering ±1 and 0 in T, we try to turn the second vector into something equally as nice. We want to work with 0's and ±1's, so perhaps we take $\vec{c} \pmod 2$? So all c_i are either 0 or 1.

Aha! If after the multiplication we have an entry with an even number of nonzero terms, we just set half to -1 and the other half to 1, and we have sum of 0 as desired.

Claim 14.2 — $\det(T) \equiv 1 \pmod 2$.

Proof. Take $T \pmod 2$. Then the determinant is the wedge product

$$(e_2 + e_3 + \dots e_{1000}) \wedge (e_1 + e_3 + \dots e_{1000}) \wedge \cdots \wedge (e_1 + e_2 + \dots e_{999})$$

Notice that in the final wedge, e_i is *never* in the ith position (this took myself some convincing, but you should be able to see that too). This is called a derangement, and is defined recursively as

$$!n = \begin{cases} 1 & \text{if } n = 0, \\ n\left(!(n-1)\right) + (-1)^n & \text{if } n > 0. \end{cases}$$

I can try and prove the recursion itself later, but even numbers always give odd number of derangements. So $!1000 \equiv 1 \pmod 2$ as desired. ■

T is invertible because its determinant can never equal 0. So that implies $\vec{c} = \vec{0}$. Contradiction! □